THE SPECTRUM OF THE CESÀRO-HARDY OPERATOR ON THE HILBERT-PÓLYA SPACE

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Abstract. By considering the spectrum of the Cesàro-Hardy operator on the Hilbert-Pólya space, we proved the Riemann hypothesis for Riemann zeta function and Dirichlet L-function.

1. Introduction

Denote $\mathbb{R}_+^\times = (0, \infty)$. Let $L^2(\mathbb{R}_+^\times)$ be the complex Hilbert space with the usual inner product, i.e.,

$$\langle f(x), g(x) \rangle = \int_0^\infty f(x)\overline{g(x)}dx.$$

Here we view $L^2(\mathbb{R}_+^\times)$ as a Hilbert space in the meaning of quotient space, i.e., each $f \in L^2(\mathbb{R}_+^\times)$ with $\int_0^\infty |f(x)|^2 dx = 0$ is equivalent to the zero function on $\mathbb{R}_+^\times$.

The Cesàro-Hardy operator $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$ is defined by

$$\mathcal{C}(f)(x) = \frac{1}{x}\int_0^x f(t)dt,$$

where $f(x) \in L^2(\mathbb{R}_+^\times)$ is a locally integrable function. Then $\mathcal{C}$ is a bounded operator on $L^2(\mathbb{R}_+^\times)$ by Hardy inequality. In [3], Brown, Halmos and Shields showed that the spectrum of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$ is the circle

$$\sigma(\mathcal{C}, L^2) = \{z \in \mathbb{C} : |1 - z| = 1\}.$$

This result has been generalized by D. W. Boyd [4] to L^p space. If we consider the operator $\mathcal{C} - 1$, then we will find that it is a unitary operator on $L^2(\mathbb{R}_+^\times)$. A well known result which says the spectrum of unitary operator is contained in the unit circle $\{z \in \mathbb{C} : |z| = 1\}$

The adjoint of the Cesàro-Hardy operator $\mathcal{C}$ is $\mathcal{C}^*$, which is defined by

$$\langle \mathcal{C}f, g\rangle = \langle f, \mathcal{C}^* g\rangle,$$

for $f, g \in L^2(\mathbb{R}_+^\times)$. The explicit form of $\mathcal{C}^*$ is

$$\mathcal{C}^* f(x) = \int_x^\infty \frac{f(t)}{t}dt.$$

Motivated by Alain Connes's spectral interpretation for the zeros of L-functions, Ralf Meyer[15] proved that the eigenvalues of the transpose D_-^t (see [21, §2.1])of the operator D_-(induced by D on some function space) acting on a nuclear Fréchet

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space are exactly the nontrivial zeros of $\zeta(s)$. Later, Xian-Jin Li [12] proved that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-. His method has been generalized to Dirichlet L-function and L-function associated with newforms by Dongsheng Wu[23]. Liming Ge, Xian-Jin Li, Dongsheng Wu and Boqing Xue in [9] proved that the correspondence between the set of eigenvalues of D_- acting on $\mathcal{H}$ and the set of nontrivial zeros of $\zeta(s)$ is one-to-one.

Inspired by the above results, we construct the Hilbert–Pólya space of operator $\mathcal{C}$. The idea to prove Rieman hypothesis is as follows: We take Riemann zeta function as an example. First, we construct an invariant space V_ζ (Definition5.2) of $\mathcal{C}$ and $\mathcal{C}^*$ in $L^2(\mathbb{R}_+^\times)$. For each nontrivial zero ρ of $\zeta(s)$, we construct a function F_ρ (Equation(5.3)). Let $\overline{V_\zeta}$ be the closure of V_ζ in $L^2(\mathbb{R}_+^\times)$. The key result is to show $F_\rho \notin \overline{V_\zeta}$ (Theorem6.15). Then the Riemann hypothesis can be deduced from the property of spectrum $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$ (Theorem7.7).

In this view, the Riemann hypothesis comes from the symmetry of the Cesàro-Hardy operator. The adjoint operator $\mathcal{C}^*$ and the operator D_- are inverse in some way, which is similar to the case of elliptic function and its inverse function. However, it is better to consider $\mathcal{C}^*$ than D_-, because the first one is bounded.

2. Some properties of operators $\mathcal{C}, \mathcal{C}^*$ and $\mathcal{Z}$

Let $C^\infty(\mathbb{R}_+^\times)$ be the set of smooth complex valued functions on $\mathbb{R}_+^\times$ and $\mathbb{N}$ be the set of nonnegative integers. The following notations are from [23]:

$$\mathcal{H}_0 = \{f \in C^\infty(\mathbb{R}_+^\times) \mid \lim_{x\to\infty} x^m f^{(n)}(x) = 0 \text{ and } f^{(n)}(0) := \lim_{x\to 0^+} f^{(n)}(x) \text{ exists}, \forall m, n \in \mathbb{N}\}.$$

$$\mathcal{H}_\cap := \{f \in \mathcal{H}_0 \mid \int_0^\infty f(x)dx = 0, f(0) = 0, \text{and } f^{(2n+1)}(0) = 0,\ \forall n \in \mathbb{N}\}.$$

$$\mathcal{H}_- := \{f \in \mathcal{H}_0 \mid f^{(n)}(0) = 0 \text{ for } n \in \mathbb{N}\}.$$

Here, the above definitions of $\mathcal{H}_\cap$ coincide with Meyer's original construction (see [23, §1.2]). In fact, if $f(x)$ is an even Schwartz function over $\mathbb{R}$, then $f^{(2n+1)}(x)$ is an odd function, hence $f^{(2n+1)}(0) = 0$.

By L'Hôspital's rule, we have

$$\lim_{x\to 0^+} x^{-m} f^{(n)}(x) = 0, \quad \forall m, n \in \mathbb{N},\ \forall f(x) \in \mathcal{H}_-.$$

Let χ be a nontrivial primitive Dirichlet character. Define

$$\mathcal{H}_\cap^\chi := \{f \in \mathcal{H}_0 \mid f^{(2n+1)}(0) = 0 \text{ if } \chi(-1) = 1, f^{(2n)}(0) = 0 \text{ if } \chi(-1) = -1, \forall n \in \mathbb{N}\}.$$

If χ be a trivial primitive Dirichlet character. Define

$$\mathcal{H}_\cap^\chi := \{f \in \mathcal{H}_0 \mid \int_0^\infty f(x)dx = 0, f(0) = f^{(2n+1)}(0) = 0, \forall n \in \mathbb{N}\}.$$

Remark 2.1. In [23], he does not distinguish the trivial character and nontrivial character. They are different for L-function. If χ is a trivial character, then the L-function has a simple pole at $s = 1$. If χ is a nontrivial character, then the L-function is an entire function. The conditions here $\int_0^\infty f(x)dx = 0$ are designed to eliminate the effects of poles of L-function when considering the Mellin transform of the function $\mathcal{Z}_\chi f$(See below for the operator $\mathcal{Z}_\chi$).

Since $\mathcal{H}_0$ is a subspace of $L^2(\mathbb{R}_+^\times)$, $\mathcal{H}_0$ is a unitary space, i.e., a complex space with inner product. We define two operators $\mathcal{D}, \mathcal{M}$ on $\mathcal{H}_0$ by

$$\mathcal{D}f(x) = -f'(x), \quad \mathcal{M}f(x) = xf(x).$$

It is easy to check that

$$\mathcal{M}\mathcal{D} - \mathcal{D}\mathcal{M} = 1.$$

For $f \in \mathcal{H}_-$, we can “formally ”defined

$$\mathcal{D}^{-1}f(x) = -\int_0^x f(t)dt, \quad \mathcal{M}^{-1}f(x) = \frac{f(x)}{x}.$$

But the action of $\mathcal{D}^{-1}$ is not closed on $\mathcal{H}_-$. We just “formally” view the operator $\mathcal{C}$ as the inverse of $-\mathcal{D}\mathcal{M}$ by

$$(-\mathcal{D}\mathcal{M})^{-1}f = \mathcal{M}^{-1}(-\mathcal{D})^{-1}f = \mathcal{M}^{-1}\int_0^x f(t)dt = \frac{1}{x}\int_0^x f(t)dt = \mathcal{C}f.$$

For $f \in \mathcal{H}_\cap$, define the operator $\mathcal{Z}$ by

$$(\mathcal{Z}f)(x) = \sum_{n=1}^\infty f(nx)$$

and for $f \in \mathcal{H}_\cap^\chi$, define the operator $\mathcal{Z}_\chi$ by

$$(\mathcal{Z}_\chi f)(x) = \sum_{n=1}^\infty \chi(n)f(nx).$$

Then we have $\mathcal{Z}\mathcal{H}_\cap, \mathcal{Z}_\chi\mathcal{H}_\cap^\chi \subset \mathcal{H}_-$(see [23, Thm.2.9], [15, Thm.3.3], §6 in [15]).

Denote

$$\eta(x) = 8\pi x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2}, \quad \text{for } \zeta(s);$$

For character χ, let

$$\eta_\chi(x) = \begin{cases} 8\pi x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = 1 \\ xe^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = -1. \end{cases}$$

Then we have $\mathcal{Z}\eta, \mathcal{Z}_\chi\eta_\chi \in \mathcal{H}_-$.

For $f(x) \in \mathcal{H}_0$, its Mellin transform is

$$\widehat{f}(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Then $\widehat{f}(s)$ admits a meromorphic extension to the whole complex plane and its only singularities are simple poles at a subset of non-positive integers(see [23, Lem2.1]).

Proposition 2.2. $\mathcal{Z}\mathcal{H}_\cap \nsubseteq \mathcal{H}_\cap$.

Proof. let $\eta(x) = 8\pi x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2} \in \mathcal{H}_\cap$. For $\mathrm{Re}(s) > 1$, considering the Mellin transformation of $\mathcal{Z}\eta$, we have

$$\widehat{\mathcal{Z}\eta}(s) = \zeta(s)\widehat{\eta}(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Since $\mathcal{Z}\eta(x) \in \mathcal{H}_-$, we have $\widehat{\mathcal{Z}\eta}(s)$ is an entire function. Hence,

$$\widehat{\mathcal{Z}\eta}(1) = \pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) = 1,$$

i.e., $\int_0^\infty \mathcal{Z}\eta(x)dx = 1$. $\square$

Remark 2.3. *A fault" proof" of Proposition2.2: First, we have $\mathcal{Z}\mathcal{H}_\cap \subseteq \mathcal{H}_-$. On the other hand, for $f \in \mathcal{H}_\cap$, we have*

$$\int_0^\infty \sum_{n=1}^\infty f(nx)dx = \sum_{n=1}^\infty \int_0^\infty f(nx)dx = \sum_{n=1}^\infty \int_0^\infty \frac{1}{n} f(x)dx = 0.$$

Hence, $\mathcal{Z}f \in \mathcal{H}_\cap$.

The main problem is that the sum and the integral does not commute in this case.

Proposition 2.4. *$\mathcal{C}\mathcal{H}_\cap \nsubseteq \mathcal{H}_\cap$.*

Proof. The action of $\mathcal{C}$ on $\mathcal{H}_\cap$ is not closed. For example, let $\eta(x) = x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2} \in \mathcal{H}_\cap$. A direct calculation shows that

$$\mathcal{C}\eta(x) = -\frac{x^2}{2}e^{-\pi x^2} \notin \mathcal{H}_\cap.$$

□

Let $\mathcal{C}^*$ be the adjoint operator of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times)$. Then

Lemma 2.5. *For $f(x) \in \mathcal{H}_0$, we have*

$$\mathcal{C}^* f(x) = \int_x^\infty \frac{f(t)}{t} dt$$

Proof. For each $f, g \in \mathcal{H}_0$, there is

$$\begin{aligned}
\langle \mathcal{C}f, g\rangle &= \int_0^\infty \frac{1}{x}\int_0^x f(t)dt \cdot \overline{g(x)}dx \\
&= \int_0^\infty \int_0^x f(t)dt \cdot \overline{\left(\frac{g(x)}{x}\right)}dx \\
&= -\int_0^\infty \int_0^x f(t)dtd\int_x^\infty \overline{\left(\frac{g(t)}{t}\right)}dt \\
&= -\int_0^x f(t)dt \cdot \int_x^\infty \overline{\left(\frac{g(t)}{t}\right)}dt\Big|_0^\infty + \int_0^\infty \int_x^\infty \overline{\left(\frac{g(t)}{t}\right)}dtd\int_0^x f(t)dt \\
&= \int_0^\infty f(x)\int_x^\infty \overline{\left(\frac{g(t)}{t}\right)}dtdx \\
&= \langle f, \mathcal{C}^* g\rangle.
\end{aligned}$$

By the definition of integral, we have

$$\overline{\mathcal{C}^* g(x)} = \int_x^\infty \overline{\left(\frac{g(t)}{t}\right)}dt = \overline{\int_x^\infty \frac{g(t)}{t}dt}.$$

Hence, $\mathcal{C}^* f(x) = \int_x^\infty \frac{f(t)}{t}dt$. □

Theorem 2.6. *The operator $\mathcal{C}^* - 1$ and $\mathcal{C} - 1$ are unitary on $L^2(\mathbb{R}_+^\times)$.*

Proof. There exist the following norm equalities(see [13, Example1.6]): For $f \in L^2(\mathbb{R}_+^\times)$,

$$\|(\mathcal{C} - 1)f\| = \|(\mathcal{C}^* - 1)f\| = \|f\|,$$

where $\mathcal{C}^* - 1 = (\mathcal{C} - 1)^*$ is the adjoint operator of $\mathcal{C} - 1$. This means the bounded operator $\mathcal{C} - 1$ and $(\mathcal{C} - 1)^*$ are isometry on $L^2(\mathbb{R}_+^\times)$. By [7, Thm4.5.15], we have $(\mathcal{C} - 1)^*(\mathcal{C} - 1) = (\mathcal{C} - 1)(\mathcal{C} - 1)^* = 1$. This means $\mathcal{C}^* - 1$ and $\mathcal{C} - 1$ are unitary. □

Corollary 2.7. *$\mathcal{C}^*$ and $\mathcal{C}$ are commutative on $L^2(\mathbb{R}_+^\times)$.*

Proof. In fact, from the equation $(\mathcal{C}-1)^*(\mathcal{C}-1) = (\mathcal{C}-1)(\mathcal{C}-1)^* = 1$, we have

$$\mathcal{C}^*\mathcal{C} = \mathcal{C}\mathcal{C}^* = \mathcal{C} + \mathcal{C}^*.$$

□

3. The Hilbert space $L^2(\mathbb{R}_+^\times, dx)$ and Hardy space $H^2(\Omega)$

For the multiplicative group $\mathbb{R}_+^\times$, the corresponding Haar measure is $\frac{dx}{x}$. Let $L^2(\mathbb{R}_+^\times, \frac{dx}{x})$ (resp. $L^2(\mathbb{R}_+^\times, dx)$) be the complex Hilbert space of square integral function on $\mathbb{R}_+^\times$ with respect to the measrue $\frac{dx}{x}$ (resp. dx).

Consider the pairing $\mathbb{R}_+^\times \times \mathbb{R}i \to S^1$, $(r, ti) \mapsto r^{-ti}$. Under this pairing, $\mathbb{R}i$ can be viewed as the character group of $\mathbb{R}_+^\times$. Denote $\widehat{\mathbb{R}}_+^\times$ the character group of $\mathbb{R}_+^\times$, i.e.,

$$\widehat{\mathbb{R}}_+^\times := \{\psi : \mathbb{R}_+^\times \to S^1 \mid \psi \text{ is continuous group homomorphism.}\}$$

A natural topology on $\widehat{\mathbb{R}}_+^\times$ is compact open topology. Under this topology, we have an topological group isomorphism

$$\widehat{\mathbb{R}}_+^\times \simeq \mathbb{R}i, \quad \psi_{ti} \mapsto ti,$$

where $\psi_{ti}(x) = x^{-ti}$ Similarly, we have $\widehat{\mathbb{R}i} \simeq \mathbb{R}_+^\times$.

Definition 3.1. (see [20, §3.3]) Let $f \in L^1(\mathbb{R}_+^\times, \frac{dx}{x})$. Then we define $\widehat{f} : \widehat{\mathbb{R}}_+^\times \to \mathbb{C}$, the Fourier transform of f, by the formula

$$\widehat{f}(\psi) = \int_{\mathbb{R}_+^\times} f(x)\overline{\psi}(x)\frac{dx}{x}.$$

Theorem 3.2. *Under the isomorphism $\widehat{\mathbb{R}}_+^\times \simeq \mathbb{R}i, \quad \psi_{ti} \mapsto ti$, the Fourier transform of $f \in L^1(\mathbb{R}_+^\times, \frac{dx}{x})$ is the Mellin transform which restricts on the line $\mathbb{R}i$.*

Proof. Since $\psi_{ti}(x) = x^{-ti}$, we have $\overline{\psi}_{ti}(x) = x^{ti}$. Denote $s = ti$. View ψ_s as s. Then Fourier transform of f is

$$\widehat{f}(\psi_s) = \int_0^\infty f(x)\overline{\psi}_s(x)\frac{dx}{x} = \int_0^\infty f(x)x^{s-1}dx.$$

This is just the Mellin transform on $\mathbb{R}i$. □

Denote $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ the multiplicative group of $\mathbb{C}$. Consider the pairing

$$\mathbb{R}_+^\times \times \mathbb{C} \to \mathbb{C}^\times, \quad (r, s) \mapsto r^{-s}.$$

We use $\widehat{\mathbb{R}_+^\times}$ denoting the set of quasi-characters of $\mathbb{R}_+^\times$. A quasi-character ϕ of $\mathbb{R}_+^\times$ is a continuous homomorphism $\phi : \mathbb{R}_+^\times \to \mathbb{C}^\times$. Since $\mathbb{R}_+^\times$ has no nontrivial compact open subgroup, each quasi-character of $\mathbb{R}_+^\times$ is *unramified* (see [14, XIV,§2]), also called *principal* by Weil(see [22, VII,§3]).

Theorem 3.3. *Each quasi-character ϕ of $\mathbb{R}_+^\times$ is of the form*

$$\phi(x) = x^{-s},$$

where s is uniquely determined by ϕ. Hence, ϕ can be written by ϕ_s Thus $\widehat{\mathbb{R}_+^\times} \simeq \mathbb{C}$.

Proof. See [14, XIV,§2,Prop.1], [22, VII,§3, Cor.1], [20, §7.1]. □

Definition 3.4. Given the pairing $\langle, \rangle : \mathbb{R}_+^\times \times \mathbb{C} \to \mathbb{C}^\times,\ (x,s) \mapsto \langle x,s\rangle = x^{-s}$. The quasi-character $\phi_s \in \widehat{\mathbb{R}_+^\times}$ is defined by

$$\phi_s(x) = \langle x, s\rangle = x^{-s}.$$

The involution ϕ_s^{-1} of ϕ_s is defined by

$$\phi_s^{-1}(x) = x^s.$$

Similarly, we have the Fourier transform for quasi-characters.

Definition 3.5. Let $f \in L^1(\mathbb{R}_+^\times, \frac{dx}{x})$ and $\widehat{\mathbb{R}_+^\times}$ be the quasi-character group. Then we define $\widehat{f} : \widehat{\mathbb{R}_+^\times} \to \mathbb{C}$, the Fourier transform of f, by the formula

$$\widehat{f}(\phi) = \int_{\mathbb{R}_+^\times} f(x)\phi^{-1}(x)\frac{dx}{x}.$$

As in Theorem3.2, we have

Theorem 3.6. *Under the isomorphism* $\widehat{\mathbb{R}_+^\times} \simeq \mathbb{C}, \quad \phi_s \mapsto s$*, the Fourier transform of* $f \in L^1(\mathbb{R}_+^\times)$ *is the Mellin transform* $\widehat{f}(s)$*, which is convergent in some region of* $\mathbb{C}$*.*

Next, we construct the Fourier inversion formula, which essentially is the inverse Mellin transform. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathbb{R}_+^\times \times \mathbb{R}i \arrow[r, "\langle,\rangle"] \arrow[d] & S^1 \arrow[d] \\
\mathbb{R}_+^\times \times (\sigma + \mathbb{R}i) \arrow[r] & \mathbb{R}_+^\times \times \mathbb{C} \arrow[r, "\langle,\rangle"] \arrow[d] & \mathbb{C}^\times. \\
 & \mathbb{R}_+^\times \times \widehat{\mathbb{R}_+^\times} \arrow[ur] &
\end{tikzcd}$$

where $\sigma \in \mathbb{R}$.

Definition 3.7. Consider the pairing $\langle, \rangle$ restricting on $\mathbb{R}_+^\times \times (\sigma + \mathbb{R}i)$. Define $\widehat{\sigma + \mathbb{R}i}$ the set of maps $\langle x, \rangle :\ \sigma + \mathbb{R}i \to \mathbb{C}^\times$, where $x \in \mathbb{R}_+^\times$. Define ${}_\sigma\widehat{\mathbb{R}}_+^\times$ the set of maps $\langle, s\rangle : \mathbb{R}_+^\times \to \mathbb{C}^\times$, where $s \in \sigma + \mathbb{R}i$.

$\widehat{\sigma + \mathbb{R}i}$ can be viewed as the line $\mathbb{R}_+^\times$, because there is a one-to-one correspondence between them. Similarly, ${}_\sigma\widehat{\mathbb{R}}_+^\times$ can be viewed as the line $\sigma + \mathbb{R}i$.

Let $V(G)$ denote the complex span of the continuous functions of positive type(see [20, §3.2]) on the locally compact group G. Define

$$V^1(G) = V(G) \cap L^1(G).$$

Theorem 3.8. (*The inverse Mellin transform*)
Consider the pairing $\langle, \rangle : \mathbb{R}_+^\times \times \mathbb{C} \to \mathbb{C}^\times$*. Let* $\phi_s = \langle, s\rangle \in \widehat{\mathbb{R}_+^\times}$ *be a quasi-character. The Haar measure on* $\widehat{\mathbb{R}_+^\times}$ *is* $d\phi$*. Denote the restriction of* ϕ *on the line* $\sigma + \mathbb{R}i$ *by* ϕ^σ*. The measure* $d\phi$ *restricting on* $\sigma + \mathbb{R}i$ *is denoted by* $d\phi^\sigma$*. Then for all* $f \in V^1(\mathbb{R}_+^\times)$*,*

$$f(x) = \int_{{}_\sigma\widehat{\mathbb{R}}_+^\times} \widehat{f}(\phi)\phi(x)d\phi^\sigma = \frac{1}{2\pi i}\int_{\sigma+\mathbb{R}i} \widehat{f}(s)x^{-s}ds.$$

If $f(x)$ is analytic on $\mathbb{R}_+^\times$ and satisfies the asymptotic conditions

$$f(x) = O(x^{-\alpha}), \quad x \to 0,$$
$$f(x) = O(x^{-\beta}), \quad x \to \infty,$$

where $\alpha < \beta$. Then the Mellin transform $\widehat{f}(s)$ is analytic in the strip $\alpha < \mathrm{Re} s < \beta$. For example, for $f(x) = \frac{1}{x+1} \in L^2(\mathbb{R}_+^\times, dx)$, its Mellin transform $\widehat{f}(s)$ is analytic in the strip $0 < \mathrm{Re}(s) < 1$; for $g(x) = \frac{\sqrt{x}}{x+1} \in L^2(\mathbb{R}_+^\times, \frac{dx}{x})$, its Mellin transform $\widehat{g}(s)$ is analytic in the strip $-\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$.

Proposition 3.9. *Denote $I = (1, \infty)$. Under the isomorphisms $e^x : \mathbb{R}_+^\times \to I$, $\log x : I \to \mathbb{R}_+^\times$, the space $L^2(\mathbb{R}_+^\times, dx)$ is isometric to $L^2(I, \frac{dx}{x})$, a subspace of $L^2(\mathbb{R}_+^\times, \frac{dx}{x})$. We denote this isometry by*

$$\mathcal{E} : L^2(\mathbb{R}_+^\times, dx) \to L^2(I, \frac{dx}{x}) \tag{3.1}$$

Proof. Let $f \in L^2(I, \frac{dx}{x})$. We can view f as an element $\widetilde{f}$ of $L^2(\mathbb{R}_+^\times, \frac{dx}{x})$ by

$$\widetilde{f} = \begin{cases} f, & \text{if } x > 1, \\ 0, & \text{if } 0 < x \leq 1. \end{cases}$$

Then $L^2(I, \frac{dx}{x})$ is a subspace of $L^2(\mathbb{R}_+^\times, \frac{dx}{x})$.

Take $g(x) \in L^2(\mathbb{R}_+^\times, dx)$ and $f(x) \in L^2(I, \frac{dx}{x})$. Then we have

$$\int_0^\infty |g(x)|^2 dx = \int_1^\infty |g(\log y)|^2 \frac{dy}{y};$$
$$\int_1^\infty |f(x)|^2 \frac{dx}{x} = \int_0^\infty |f(e^y)|^2 dy.$$

The above equalities show that $L^2(\mathbb{R}_+^\times, dx)$ is isometric $L^2(I, \frac{dx}{x})$. $\square$

Denote the strip $\Omega_{(0,\frac{1}{2})} := \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < \frac{1}{2}\}$. If there is no confusion, we write Ω for $\Omega_{(0,\frac{1}{2})}$. Denote the half-plane

$$\Omega_{>0} = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z)\},$$
$$\Omega_{<\frac{1}{2}} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < \frac{1}{2}\}.$$

The Hardy space for up half-plane is classical. The summary of basic properties for Hardy space for up half-plane can be found in [2]. The theory for right or left half-plane is similar, because these planes can be obtained from half-plane by times $-i$ or i. Recall that the Hardy space $H^2(\Omega_{>0})$ for half-plane is the space of analytic function $f : \Omega_{>0} \to \mathbb{C}$, for which

$$\| f \|_{H^2(\Omega_{>0})} = \sup_{0<x} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |f(x+yi)|^2 dy \right)^{\frac{1}{2}} < \infty.$$

For convenience, we use the notation $\| f \|_{\Omega_{>0}}$ for $\| f \|_{H^2(\Omega_{>0})}$.

Suppose $f \in H^2(\Omega_{>0})$. Then f satisfies the growth condition

$$|f(z)|^2 \leq \frac{C \| f \|_{\Omega_{>0}}^2}{\mathrm{Re}(z)}, \quad z \in \Omega_{>0},$$

where C is the constant. The limit $\lim\limits_{x\to 0} f(x+yi)$ exists for almost every y in $\mathbb{R}$, and we may define the boundary function on $\mathbb{R}i$, denoted by f^*, i.e.,

$$f^*(z) = \lim_{x\to 0} f(x+yi).$$

This function is square-integrable and

$$\| f \|^2_{\Omega_{>0}} = \frac{1}{i}\int_{-\infty i}^{+\infty i} |f^*(z)|^2 dz$$

Then we have an isometry

$$\iota : H^2(\Omega_{>0}) \to L^2(\mathbb{R}i),\ f \mapsto \iota(f) = f^*.$$

The Hardy space for $H^2(\Omega_{<\frac{1}{2}})$ is similar.

Let $\mathscr{M}^{-1}$ be the inverse Mellin transform and $\mathscr{F}$ be the Fourier transform. We have the diagram

$$\begin{tikzcd}
L^2(\mathbb{R}i) \arrow[d, shift left, "\mathscr{F}^{-1}"] & H^2(\Omega_{>0}) \arrow[l, hook', "\iota"'] \arrow[d, hook, "\cap"] & \\
L^2(\mathbb{R}_+^\times, \frac{dx}{x}) \arrow[u, shift left, "\mathscr{F}"] & H^2(\Omega) \arrow[ul, "\iota"] \arrow[dl, dashed, "\mathscr{M}^{-1}"'] \arrow[dr, "\iota"] & H^2(\Omega_{<\frac{1}{2}}) \arrow[l, hook', "\supset"'] \arrow[d, hook, "\iota"] \\
L^2(\mathbb{R}_+^\times, dx) \arrow[u, hook, "\mathcal{E}"] & & L^2(\frac{1}{2}+\mathbb{R}i)
\end{tikzcd}$$

The map

$$H^2(\Omega) \overset{\mathscr{M}^{-1}}{\dashrightarrow} L^2(\mathbb{R}_+^\times, dx)$$

will be studied in the next section.

Theorem 3.10. *Let $f \in H^2(\Omega_{>0})$. Then $f \in H^2(\Omega)$. Moreover, $\| f \|^2_{\Omega_{>0}} = \| f \|^2_{\Omega}$.*

Proof. Since $f \in H^2(\Omega_{>0})$, the norm equality is obtained by

$$\| f^* \|_{L^2(\mathbb{R}i)} = \| f \|^2_{\Omega_{>0}} \geq \| f \|^2_{\Omega} \geq \| f^* \|_{L^2(\mathbb{R}i)}\ .$$

The last inequality is from [24, Thm.2]. ☐

Theorem 3.11. *The Fourier transform $\mathscr{F} : L^2(\mathbb{R}_+^\times, \frac{dx}{x}) \to L^2(\mathbb{R}i)$ and the inverse Fourier transform $\mathscr{F}^{-1} : L^2(\mathbb{R}i) \to L^2(\mathbb{R}_+^\times, \frac{dx}{x})$ are isometries.*

Proof. Since $\mathbb{R}_+^\times$ and $\mathbb{R}i$ are dual to each other, the theorem follows from [20, Thm.3-26]. ☐

Theorem 3.12. *The Hardy space $H^2(\Omega_{>0})$ is isometric to a subspace of $L^2(\mathbb{R}_+^\times, \frac{dx}{x})$ by $\mathscr{F}^{-1}\iota$.*

4. Theorem of Paley and Wiener for Mellin transform

The theorem of Paley and Wiener for holomorphic Fourier transform constructs unitary operator between $L^2(\mathbb{R}_+^\times, dx)$ and the Hardy space $H^2(\mathbb{H})$, where $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the upper half-plane(see [19, §19.1-2]). Its explicit form can

be found in [19, Thm.19.2], which says that there exist a surjective isometry from $L^2(\mathbb{R}_+^\times, dx)$ to $H^2(\mathbb{H})$. We write this isometry by

$$\mathscr{F} : L^2(\mathbb{R}_+^\times, dx) \to H^2(\mathbb{H}), \quad F(x) \mapsto \mathscr{F}F(z) = \int_0^\infty F(x) e^{ixz} dx \qquad (z \in \mathbb{H}).$$

Denote $\Omega_{<0}$ the left half-plane of $\mathbb{C}$. Then the canonical isometry between $H^2(\mathbb{H})$ and $H^2(\Omega_{<0})$ is

$$\mathcal{I} : H^2(\mathbb{H}) \to H^2(\Omega_{<0}),\ f(z) \mapsto f(-is).$$

The canonical isometry between $H^2(\mathbb{H})$ and $H^2(\Omega_{>0})$ is

$$\mathcal{I} : H^2(\mathbb{H}) \to H^2(\Omega_{>0}),\ f(z) \mapsto f(is).$$

The integral is

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f(x+ia)|^2 dx &= \int_{-\infty+ai}^{+\infty+ai} |f(z)|^2 dz \\
&= \int_{a+i\infty}^{a-i\infty} |f(is)|^2 d(is) \quad (\text{where } z = is) \\
&= \frac{1}{i} \int_{a-i\infty}^{a+i\infty} |f(is)|^2 ds
\end{aligned}$$

We have the following theorem commutative diagram

Theorem 4.1. *There is a commutative diagram as follows*

$$\begin{tikzcd}
H^2(\mathbb{H}) \arrow[r, "\mathcal{I}"] \arrow[d, "\mathscr{F}^{-1}"] & H^2(\Omega_{<0}) \arrow[dr, "\iota"] & \\
L^2(\mathbb{R}_+^\times, dx) \arrow[r, "\mathcal{E}"] & L^2(\mathbb{R}_+^\times, \frac{dx}{x}) \arrow[r, "\mathscr{M}"] & L^2(\mathbb{R}i).
\end{tikzcd}$$

Proof. For $f(z) \in H^2(\mathbb{H})$, there exists an $F(x) \in L^2(\mathbb{R}_+^\times, dx)$ such that

$$f(z) = \int_0^\infty F(t) e^{izt} dt.$$

Thus $\mathscr{F}^{-1} f = F(t)$.

Since

$$\mathcal{E}(F(t)) = \begin{cases} F(\log x), & \text{if } x > 1 \\ 0, & \text{otherwise,} \end{cases}$$

one has $\mathscr{M}(\mathcal{E}(F(t))) = \int_1^\infty F(\log x) x^{s-1} dx$, where $s \in \mathbb{R}i$.

Let $iz = s$. We have, for $s \in \Omega_{<0}$,

$$\begin{aligned}
\mathcal{I}(f)(s) &= f(-is) \\
&= \int_0^\infty F(t) e^{st} dt \\
&= \int_1^\infty F(\log y) y^s d\log y \\
&= \int_1^\infty F(\log y) y^{s-1} dy.
\end{aligned}$$

Thus $\iota\mathcal{I}(f) = \lim_{x\to 0^-}\int_1^\infty F(\log y)y^{s-1}dy$, where $s = x+iy$. Since for $s \in \Omega_{<0}$,

$$\iota\mathcal{I}(f) = \int_1^\infty F(\log y)y^{s-1}dy = \mathscr{M}\mathcal{E}\mathscr{F}^{-1}(f)$$

Thus they are equal on $s \in \mathbb{R}i$. Hence, we get the commutative diagram. □

Similar to Theorem4.1, we have

Theorem 4.2. *There is a commutative diagram as follows*

$$\begin{tikzcd}
H^2(\mathbb{H}) \arrow[r, "\mathcal{I}"] \arrow[d, "\mathscr{F}^{-1}"] & H^2(\Omega_{>0}) \arrow[dr, "\iota"] & \\
L^2(\mathbb{R}_+^\times, dx) \arrow[r, "\mathcal{E}"] & L^2(\mathbb{R}_+^\times, \frac{dx}{x}) \arrow[r, "\mathscr{M}"] & L^2(\mathbb{R}i).
\end{tikzcd}$$

We follow the theorem of Paley and Wiener to prove the case of Mellin transform.

Theorem 4.3. *Denote $\Omega_a = \{z \in \mathbb{C} \mid 0 < Re(z) < a\}$, where $a \le \infty$. Let $H^2(\Omega_a)$ be the Hardy space on Ω_a and*

$$\sup_{0<x<a}\frac{1}{2\pi}\int_{-\infty}^{+\infty}|f(x+iy)|^2dy = C < \infty.$$

Then there exists an $F \in L^2(\mathbb{R}_+^\times, dx)$ such that

$$f(s) = \int_0^\infty F(x)x^{s-1}dx, \quad s \in \Omega_a,$$

and

$$\int_0^\infty |F(x)|^2dx \le C.$$

If $a = \infty$, we have $\int_0^\infty |F(x)|^2dx = C$.

Proof. Fix $x, 0 < x < a$. Take a constant $c \in (0,a)$. For each $\alpha > 0$, let Γ_α be the rectangular path with vertices at $c \pm \alpha i$ and $x \pm \alpha i$. By Cauchy's theorem, we have

$$\int_{\Gamma_\alpha} f(s)t^{-s}ds = 0, \tag{4.1}$$

where $t \in \mathbb{R}_+^\times$.

Let I be the interval

$$I = \begin{cases}[c,x], & \text{if } c < x\\ [x,c], & \text{if } x < c.\end{cases}$$

For $\beta \in \mathbb{R}$, denote $\Phi(\beta)$ the integral

$$\Phi(\beta) = \int_{c+i\beta}^{x+i\beta} f(s)t^{-s}ds.$$

Then

$$|\Phi(\beta)|^2 = \left|\int_I f(u+i\beta)t^{-(u+i\beta)}du\right|^2 \le \int_I |f(u+i\beta)|^2du \cdot \int_I t^{-2u}du. \tag{4.2}$$

Let

$$\Lambda(\beta) = \int_I |f(u+i\beta)|^2du.$$

Since $\sup\limits_{0<x<a} \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f(x+iy)|^2 dy = C < \infty$, by Fubini's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \Lambda(\beta) d\beta \leq C \cdot |c-x|.$$

Hence, there is a sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and

$$\Lambda(\alpha_j) + \Lambda(-\alpha_j) \to 0, \quad (j \to \infty).$$

By equation(4.2), this shows that

$$\Phi(\alpha_j) \to 0, \quad \Phi(-\alpha_j) \to 0, \qquad (\text{as } j \to \infty). \tag{4.3}$$

Note that this holds for every $t \in \mathbb{R}_+^\times$ and the sequence $\{\alpha_j\}$ doesn't depend on t.

Define

$$g_j(x,t) = \frac{1}{2\pi i}\int_{-\alpha_j}^{\alpha_j} f(x+iy) t^{-yi} dyi.$$

Then by equations(4.1),(4.3), we deduce that

$$\lim_{j\to\infty} [t^{-x} g_j(x,t) - t^{-c} g_j(c,t)] = 0, \quad (t \in \mathbb{R}_+^\times). \tag{4.4}$$

Write $f_x(y) = f(x+iy)$. Then $f_x \in L^2(\mathbb{R})$. The Plancherel theorem for locally compact group(see [20, Thm.3-26]) asserts that

$$\lim_{j\to\infty} \int_{-\infty}^{+\infty} |\widehat{f_x}(t) - g_j(x,t)|^2 dt = 0.$$

where

$$\widehat{f_x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_x(y) t^{-yi} dy$$

is the Fourier transform of f_x about the pairing

$$\mathbb{R} \times \mathbb{R}_+^\times \to S^1, \quad (x,t) \mapsto t^{ix}.$$

Then for almost all t, a subsequence of $\{g_j(x,t)\}$ converges pointwise to $\widehat{f_x}(t)$ ([19, Thm.3.12]). Define

$$F(t) = t^{-c}\widehat{f_c}(t). \tag{4.5}$$

Then by (4.4), we have

$$F(t) = t^{-x}\widehat{f_x}(t). \tag{4.6}$$

Note that (4.5) does not involve x and that (4.6) holds for every $x \in (0,a)$. Thus

$$F(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} f(s) t^{-s} ds.$$

This is just the inverse Mellin transform of $f(s) \in H^2(\Omega_a)$. Then the Mellin transform of $F(t)$ gives

$$f(s) = \int_0^\infty F(x) x^{s-1} dx, \quad s \in \Omega_a.$$

By Plancherel theorem for locally compact group, one has

$$\int_0^\infty t^{2x}|F(t)|^2 dt = \int_0^\infty |\widehat{f_x}(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f_x(y)|^2 dy \leq C. \tag{4.7}$$

Let $x \to 0$. One obtains,

$$\int_0^\infty |F(t)|^2 dt \leq C.$$

When $a = \infty$, the equality $\int_0^\infty |F(t)|^2 dt = C$ can be obtained by the commutative diagram in Theorem4.2, in which all the maps are isometries. $\square$

Corollary 4.4. *For each Hardy space $H^2(\Omega_a)$, there is an injection*

$$\mathscr{M}^{-1} : H^2(\Omega_a) \to L^2(\mathbb{R}_+^\times, dx).$$

Moreover, $\mathscr{M}^{-1}$ is a bounded operator.

Proof. Let $f \in H^2(\Omega_a)$. Then $F(t) = \mathscr{M}^{-1} f$. Since

$$\|\mathscr{M}^{-1} f\|^2 = \|F(t)\|^2 \leq C = \|f\|^2,$$

we have $\mathscr{M}^{-1}$ is a bounded operator. $\square$

5. An invariant space of $\mathcal{C}$ and $\mathcal{C}^*$

Let H be a Hilbert space and A is a bounded operator on H. Suppose V is an invariant closed subspace of A in H. We have a canonical decomposition (see [7, Thm.3.6.6]):

$$H = V \oplus V^\perp,$$

where $V^\perp$ is the orthogonal complement of V in H. Of course, we have a canonical isomorphism

$$H/V \cong V^\perp.$$

However, there is a big difference between H/V and $V^\perp$, that is, H/V is an invariant space of A but $V^\perp$ is not an invariant space of A in general(see [10]). The properties of a morphism of *quotient Hilbert space* has been studied in [16].

Theorem 5.1. *Let $\mathcal{C}^*$ be the adjoint of Cesàro-Hardy operator on the Hilbert space $H = L^2(\mathbb{R}_+^\times)$. Suppose $V \neq H$ is an invariant subspace of $\mathcal{C}^*$. Denote $\overline{\mathcal{C}^*}$ the operator on the quotient space H/V induced by $\mathcal{C}^*$. If $V^\perp$ is an invariant subspace of $\mathcal{C}^*$, then $\overline{\mathcal{C}^*} - 1$ is a unitary operator on H/V.*

Proof. First, V is also an invariant subspace of $\mathcal{C}^* - 1$. Moreover, under the canonical isomorphism

$$H/V \simeq V^\perp, \ x + V \mapsto x_v^\perp,$$

H/V is a Hilbert space. Here, $x_v^\perp$ is the projection of x on $V^\perp$, and it does not depend on the choice $x \in x + V$. Let $x = x_v + x_v^\perp$. Then we have

$$(\mathcal{C}^* - 1)x = (\mathcal{C}^* - 1)x_v + (\mathcal{C}^* - 1)x_v^\perp.$$

Since $V^\perp$ is an invariant subspace of $\mathcal{C}^*$, it is also an invariant subspace of $\mathcal{C}^* - 1$. Thus $(\mathcal{C}^* - 1)x_v^\perp \in V^\perp$. Therefore, we have the norm equalities

$$\begin{aligned}
\| x + V \| &= \| x_v^\perp \| && \text{by definition} \\
&= \| (\mathcal{C}^* - 1)x_v^\perp \| && \text{by isometry} \\
&= \| (\mathcal{C}^* - 1)x_v^\perp + V \| && \text{since } (\mathcal{C}^* - 1)x_v^\perp \in V^\perp \\
&= \| (\mathcal{C}^* - 1)x + V \|
\end{aligned}$$

This means $\overline{\mathcal{C}^*} - 1$ is isometric on H/V, i.e., $\overline{\mathcal{C}^*} - 1$ is injective(See [7, Thm.4.5.15]). Consider the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\mathcal{C}^*-1} & H \\ \downarrow & & \downarrow \\ H/V & \xrightarrow{\overline{\mathcal{C}^*}-1} & H/V. \end{array}$$

From the commutative diagram, $\overline{\mathcal{C}^*} - 1$ is surjective, hence $\overline{\mathcal{C}^*} - 1$ is unitary. $\square$

Denote

$$\eta(x) = 8\pi x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2}, \quad \text{for } \zeta(s); \tag{5.1}$$

For character χ, let

$$\eta_\chi(x) = \begin{cases} 8\pi x^2(\pi x^2 - \frac{3}{2})e^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = 1 \\ xe^{-\pi x^2}, & \text{for } L(\chi, s) \text{ when } \chi(-1) = -1. \end{cases} \tag{5.2}$$

Because $\mathcal{C}$ and $\mathcal{C}^*$ are commutative, we have the following definition

Definition 5.2. Define the subspace V_ζ for $\zeta(s)$ which is linearly generated over $\mathbb{C}$ by

$$\{\mathcal{C}^m\mathcal{C}^{*n}\mathcal{Z}\eta \mid m, n \in \mathbb{N}, m + n \neq 0\};$$

the subspace $V_\zeta^{\mathcal{C}^*}$ is linearly generated over $\mathbb{C}$ by $\{\mathcal{C}^{*n}\mathcal{Z}\eta \mid n \in \mathbb{N}, n \neq 0\}$, and the subspace $V_\zeta^{\mathcal{C}}$ is linearly generated over $\mathbb{C}$ by $\{\mathcal{C}^m\mathcal{Z}\eta \mid m \in \mathbb{N}, m \neq 0\}$.

Define the subspace V_χ for $L(s, \chi)$ which is linearly generated over $\mathbb{C}$ by

$$\{\mathcal{C}^m\mathcal{C}^{*n}\mathcal{Z}_\chi\eta_\chi \mid m, n \in \mathbb{N}, m + n \neq 0\};$$

$V_\chi^{\mathcal{C}^*}$ is linearly generated over $\mathbb{C}$ by $\{\mathcal{C}^{*n}\mathcal{Z}_\chi\eta_\chi \mid n \in \mathbb{N}, n \neq 0\}$, and $V_\chi^{\mathcal{C}}$ is linearly generated over $\mathbb{C}$ by $\{\mathcal{C}^m\mathcal{Z}_\chi\eta_\chi \mid m \in \mathbb{N}, m \neq 0\}$.

Proposition 5.3. *The spaces V_ζ, V_χ are invariant spaces of $\mathcal{C}$ and $\mathcal{C}^*$. Moreover,*

$$V_\zeta = V_\zeta^{\mathcal{C}} + V_\zeta^{\mathcal{C}^*}, \quad V_\chi = V_\chi^{\mathcal{C}} + V_\chi^{\mathcal{C}^*}.$$

Proof. The first statement is clear, this is because $\mathcal{C}$ and $\mathcal{C}^*$ are commutative. We prove these equations by induction on the monomial term $\mathcal{C}^m\mathcal{C}^{*n}$. First, $\mathcal{C}\mathcal{C}^* = \mathcal{C} + \mathcal{C}^*$. Suppose that for $m + n \leq N - 1$,

$$\mathcal{C}^m\mathcal{C}^{*n} = f(\mathcal{C}) + g(\mathcal{C}^*),$$

where $f(X), g(X)$ are polynomials of the form $f(X) = \sum_{i=1}^{m} a_i X^i$, $g(X) = \sum_{j=1}^{n} b_j X^j$. Then when $m + n = N$, we have

$$\mathcal{C}^m\mathcal{C}^{*n} = \mathcal{C}\mathcal{C}^{m-1}\mathcal{C}^{*n} = \mathcal{C}(f_1(\mathcal{C}) + g_1(\mathcal{C}^*)),$$

where $\deg f_1(X) \leq m - 1$, $\deg g_1(X) \leq n$. Using the induction again, we have $\mathcal{C}g_1(\mathcal{C}^*) = a_1\mathcal{C} + g(\mathcal{C}^*)$. Denote $f(X) = Xf_1(X) + a_1X$. Therefore,

$$\mathcal{C}^m\mathcal{C}^{*n} = f(\mathcal{C}) + g(\mathcal{C}^*).$$

Thus the equations for V_ζ, V_χ follow from the above equality. $\square$

Remark 5.4. $\mathcal{C}$ may be irreducible on some Hilbert space H, that is, there are no nontrivial closed subspaces M of H such that $CM \subseteq M$ and $C^*M \subseteq M$. For example, see [18, §12].

Let ρ be a nontrivial zero of $\zeta(s)$ (resp. $L(\chi, s)$ with character χ). Denote

$$\begin{cases} F_\rho(x)_\zeta = \int_1^\infty \mathcal{Z}\eta(tx)t^{\rho-1}dt, & \text{if } \zeta(\rho) = 0 \\ F_\rho(x)_\chi = \int_1^\infty \mathcal{Z}_\chi\eta_\chi(tx)t^{\rho-1}dt, & \text{if } L(\chi,\rho) = 0. \end{cases} \tag{5.3}$$

Then

$$F_\rho(x)_\zeta = \int_1^\infty \mathcal{Z}\eta(tx)t^{\rho-1}dt = x^{-\rho}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt.$$

It is easy to see

$$F_\rho(x)_\zeta = x^{-\rho}\mathcal{C}^*(x^\rho \mathcal{Z}\eta).$$

Lemma 5.5. *Let $f \in C^\infty(\mathbb{R}_+^\times)$. Denote $F(x) = -xf'(x)$. Suppose $f(x)$ decays rapidly when $x \to \infty$ and $f(x) = O((\log x)^n)$ $(n \in \mathbb{N})$ when $x \to 0$. Let $\widehat{F}(s)$ be the Mellin transform of $F(x)$. For the operator $\mathcal{C}^*$, when $\mathrm{Re}(s) > 0$, there is*

$$\widehat{\mathcal{C}^*F}(s) = \frac{\widehat{F}(s)}{s}.$$

Denote $G(x) = (xf(x))'$. Suppose $f(x) = O\left(\frac{(\log x)^n}{x}\right)$ when $x \to \infty$ and $f(0) = 0$ when $x \to 0$. Let $\widehat{G}(s)$ be the Mellin transform of $G(x)$. For the operator $\mathcal{C}$, when $0 < \mathrm{Re}(s) < 1$, we have

$$\widehat{\mathcal{C}G}(s) = \frac{\widehat{G}(s)}{1-s}.$$

Proof. First, $\mathcal{C}^*F(x) = \int_x^\infty \frac{F(t)}{t}dt = \int_x^\infty \frac{-tf'(t)}{t}dt = f(x)$. Then $\widehat{\mathcal{C}^*F}(s) = \widehat{f}(s)$. On the other hand, for $\mathrm{Re}(s) > 0$,

$$\begin{aligned} \widehat{F}(s) &= \int_0^\infty F(x)x^{s-1}dx \\ &= -\int_0^\infty x^s f'(x)dx \\ &= -\int_0^\infty x^s df(x) \\ &= -x^s f(x)\Big|_0^\infty + s\int_0^\infty f(x)x^{s-1}dx \\ &= s\widehat{f}(s). \end{aligned}$$

Thus $\widehat{\mathcal{C}^*F}(s) = \frac{\widehat{F}(s)}{s}$.

It is easy to see $\mathcal{C}G(x) = f(x)$. Therefore, $\widehat{\mathcal{C}G}(s) = \widehat{f}(s)$. Moreover, when $0 < \mathrm{Re}(s) < 1$,

$$\begin{aligned}
\widehat{G}(s) &= \int_0^\infty G(x)x^{s-1}dx \\
&= \int_0^\infty x^{s-1}d(xf(x)) \\
&= x^s f(x)\Big|_0^\infty - \int_0^\infty xf(x)dx^{s-1} \\
&= (1-s)\int_0^\infty f(x)x^{s-1}dx \\
&= (1-s)\widehat{f}(s).
\end{aligned}$$

Hence, $\widehat{\mathcal{C}G}(s) = \frac{\widehat{G}(s)}{1-s}$. □

Lemma 5.6. *For positive integer j, $\mathcal{C}^{*j}\mathcal{Z}\eta$ decays rapidly when $x \to \infty$ and $|\mathcal{C}^{*j}\mathcal{Z}\eta| = O((-\log x)^{j-1})$ when $x \to 0$. However, $|\mathcal{C}^j\mathcal{Z}\eta| = O\left(\frac{(\log x)^{j-1}}{x}\right)$ when $x \to \infty$ and $\mathcal{C}^j\mathcal{Z}\eta(0) = 0$ when $x \to 0$.*

Proof. First, $\mathcal{Z}\eta \in \mathcal{H}_-$. Suppose $\mathcal{C}^{*j-1}\mathcal{Z}\eta$ decays rapidly when $x \to \infty$. Then, by induction,

$$\begin{aligned}
\lim_{x\to\infty} x^n\mathcal{C}^{*j}\mathcal{Z}\eta &= \lim_{x\to\infty} \frac{\int_x^\infty \frac{\mathcal{C}^{*j-1}\mathcal{Z}\eta}{t}dt}{x^{-n}} \\
&= \lim_{x\to\infty} \frac{-\mathcal{C}^{*j-1}\mathcal{Z}\eta(x)}{-nx^{-n}} \\
&= 0.
\end{aligned}$$

Hence, $\mathcal{C}^{*j}\mathcal{Z}\eta$ decays rapidly when $x \to \infty$.

When $x \to 0$, $\mathcal{C}^*\mathcal{Z}\eta(0) = \int_0^\infty \frac{\mathcal{Z}\eta}{t}dt$ is finite. Suppose $|\mathcal{C}^{*j}\mathcal{Z}\eta| \le -M(\log x)^{j-1}$ for sufficiently small x and for some positive constant. Then for sufficiently small $c > 0$,

$$\begin{aligned}
\left|\mathcal{C}^{*j+1}\mathcal{Z}\eta(x)\right| &\le \int_x^\infty \left|\frac{\mathcal{C}^{*j}\mathcal{Z}\eta}{t}\right| dt \\
&= \int_x^c \left|\frac{\mathcal{C}^{*j}\mathcal{Z}\eta}{t}\right| dt + \int_c^\infty \left|\frac{\mathcal{C}^{*j}\mathcal{Z}\eta}{t}\right| dt \\
&\le \int_x^c \frac{M(\log t)^{j-1}}{t}dt + \int_c^\infty \left|\frac{\mathcal{C}^{*j}\mathcal{Z}\eta}{t}\right| dt \\
&= -\frac{M}{j}(\log x)^j + \frac{M}{j}(\log c)^j + \int_c^\infty \left|\frac{\mathcal{C}^{*j}\mathcal{Z}\eta}{t}\right| dt.
\end{aligned}$$

Thus, when $x \to 0$, $|\mathcal{C}^{*j}\mathcal{Z}\eta| = O((-\log x)^{j-1})$.

By L'Hôspital's rule, it is easy to see $\mathcal{C}^j\mathcal{Z}\eta(0) = 0$ when $x \to 0$. When $x \to \infty$, $|\mathcal{C}\mathcal{Z}\eta| \le \frac{1}{x}\int_0^x |\mathcal{Z}\eta|dt \le \frac{1}{x}\int_0^\infty |\mathcal{Z}\eta|dt$. Suppose $|\mathcal{C}^j\mathcal{Z}\eta| \le M\frac{(\log x)^{j-1}}{x}$ for some

positive constant M and for sufficiently large x. Then for sufficiently large $N > 0$,

$$\begin{aligned}
\left|\mathcal{C}^{j+1}\mathcal{Z}\eta(x)\right| &\le \frac{1}{x}\int_0^x \left|\mathcal{C}^j\mathcal{Z}\eta\right| dt \\
&= \frac{1}{x}\int_0^N \left|\mathcal{C}^j\mathcal{Z}\eta\right| dt + \frac{1}{x}\int_N^x \left|\mathcal{C}^j\mathcal{Z}\eta\right| dt \\
&\le \frac{1}{x}\int_0^N \left|\mathcal{C}^j\mathcal{Z}\eta\right| dt + \frac{1}{x}\int_N^x M\frac{(\log t)^{j-1}}{t} dt \\
&= \frac{1}{x}\int_0^N \left|\mathcal{C}^j\mathcal{Z}\eta\right| dt + \frac{M}{j}\frac{(\log x)^j}{x} - \frac{M}{j}\frac{(\log N)^j}{x}.
\end{aligned}$$

Thus, $|\mathcal{C}^j\mathcal{Z}\eta| = O\left(\frac{(\log x)^{j-1}}{x}\right)$ when $x \to \infty$. □

Theorem 5.7. *Let ρ be a nontrivial zero of $\zeta(s)$ (resp. $L(\chi, s)$ with character χ). The function $F_\rho(x)$ is as in equation(5.3). Then $F_\rho(x) \notin V_\zeta$ (resp. $F_\rho(x) \notin V_\chi$).*

Proof. We prove the theorem for $\zeta(s)$. The other case is similar. Suppose $F_\rho(x) \in V_\zeta$. Then by Proposition5.3, $F_\rho(x)$ can be expressed by

$$F_\rho(x) = \sum_{j=1}^m a_j\mathcal{C}^{*j}\mathcal{Z}\eta + \sum_{k=1}^n b_k\mathcal{C}^k\mathcal{Z}\eta,$$

where $a_j, b_k \in \mathbb{C}$. Consider its Mellin transform. By Lemmas5.5,5.6, when $0 < \mathrm{Re}(s) < 1$, we have

$$\begin{aligned}
\widehat{F}_\rho(s) &= \sum_{j=1}^m a_j\widehat{\mathcal{C}^{*j}\mathcal{Z}\eta} + \sum_{k=1}^n b_k\widehat{\mathcal{C}^k\mathcal{Z}\eta} \\
&= \sum_{j=1}^m a_j\frac{\widehat{\mathcal{Z}\eta}}{s^j} + \sum_{k=1}^n b_k\frac{\widehat{\mathcal{Z}\eta}}{(1-s)^k} \\
&= \widehat{\mathcal{Z}\eta}\left(\sum_{j=1}^m \frac{a_j}{s^j} + \sum_{k=1}^n \frac{b_k}{(1-s)^k}\right)
\end{aligned}$$

Since $\widehat{\mathcal{Z}\eta} = s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is entire and $\widehat{\mathcal{Z}\eta}(0) = \widehat{\mathcal{Z}\eta}(1) = 1$, the sum $\widehat{\mathcal{Z}\eta}\left(\sum_{j=1}^m \frac{a_j}{s^j} + \sum_{k=1}^n \frac{b_k}{(1-s)^k}\right)$ is meromorphic function on $\mathbb{C}$ with at least a pole at $s = 0, 1$.

Since

$$\begin{aligned}
-xF_\rho'(x) &= -x\left(-\rho x^{-\rho-1}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt - x^{-\rho}\mathcal{Z}\eta(x)x^{\rho-1}\right) \\
&= \rho x^{-\rho}\int_x^\infty \mathcal{Z}\eta(t)t^{\rho-1}dt + \mathcal{Z}\eta(x) \\
&= \rho F_\rho(x) + \mathcal{Z}\eta(x),
\end{aligned}$$

the Mellin transform is

$$\widehat{F}_\rho(s) = \frac{\widehat{\mathcal{Z}\eta}(s)}{s-\rho} = \frac{s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)}{s-\rho}.$$

Thus, $\widehat{F}_\rho(s)$ is a holomorphic function on $\mathbb{C}$. This is a contradiction. Hence, $F_\rho(x) \notin V_\zeta$. □

6. The Hilbert-Pólya spaces

We can intuitively see that for each $f(x) \in V_\zeta$ or V_χ, its Mellin transform $\widehat{f}(s)$ has a pole at $s = 0, 1$. If $\overline{f}(x) \in \overline{V_\zeta}$, then there exist a convergent sequence $\{f_n(x)\} \in V_\zeta \subset L^2(\mathbb{R}_+^\times, dx)$ such that $\lim\limits_{n\to\infty} f_n(x) = \overline{f}(x)$ and its Mellin transform $\widehat{\overline{f}}(s)$ should have singularities at $s = 0, 1$. Since each $\widehat{f_n}(s)$ has poles at $s = 0, 1$, we expect that its limit also has poles or essential singularities at $s = 0, 1$. When we talk about limits, we should put the sequence $\{\widehat{f_n}(s)\}$ into a topological space. Some suitable topological spaces are Hardy space. We can also view $\widehat{f_n}(s)$ as elements in the formal power series field $\mathbb{C}[[s, \frac{1}{s}]]$. Then the sequence $\{\widehat{f_n}(s)\}$ in these space should have a limit. Now let's implement those ideas.

For each $0 < \varepsilon < 1$, denote the strip

$$\Delta_\varepsilon = \{z \in \mathbb{C} : \ \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}.$$

Let $H^2(\Delta_\varepsilon)$ be the Hardy space on the strip.

Theorem 6.1. *For each $0 < \varepsilon < 1$, the Mellin transform*

$$\widehat{\mathcal{Z}\eta}(s), \widehat{\mathcal{Z}_\chi \eta_\chi}(s) \in H^2(\Delta_\varepsilon),$$

where $\mathcal{Z}\eta, \mathcal{Z}_\chi\eta_\chi$ are as in §5.

Proof. We do the case for $\mathcal{Z}\eta$. The case for $\mathcal{Z}_\chi\eta_\chi$ is similar. Since $\mathcal{Z}\eta \in \mathcal{H}_- \subset \mathcal{H}_0$, we have $\widehat{\mathcal{Z}\eta}(s)$ is essentially bounded function over $\mathbb{C}$(see [23, Thm.2.2]). Then there exists a constant $c_1 > 0$ such that on the region $\Delta_\varepsilon \cap \{z \in \mathbb{C} : |\mathrm{Im}(z)| \geq 1\}$ one has

$$|s\widehat{\mathcal{Z}\eta}(s)|^2 \leq c_1.$$

On the other hand, $\widehat{\mathcal{Z}\eta}$ is analytic. Hence, $|s\widehat{\mathcal{Z}\eta}(s)|^2$ is bounded on the region $\Delta_\varepsilon \cap \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq 1\}$. Thus there exists a constant $c_2 > 0$ such that

$$|s\widehat{\mathcal{Z}\eta}(s)|^2 \leq c_2$$

on the strip Δ_ε.

For each $\sigma \in (\varepsilon, 1-\varepsilon)$, one has

$$\begin{aligned}
\frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} |\widehat{\mathcal{Z}\eta}(s)|^2 ds &\leq \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{c_2}{|s|^2} ds \\
&= c_2 \int_{-\infty}^{+\infty} \frac{1}{\sigma^2 + y^2} dy \\
&\leq c_2 \int_{-\infty}^{+\infty} \frac{1}{\varepsilon^2 + y^2} dy \\
&= \frac{c_2 \pi}{\varepsilon}.
\end{aligned}$$

Thus $\sup\limits_{\varepsilon<\sigma<1-\varepsilon} \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} |\widehat{\mathcal{Z}\eta}(s)|^2 ds < \infty$. Therefore, $\widehat{\mathcal{Z}\eta}(s) \in H^2(\Delta_\varepsilon)$. □

Theorem 6.2. *Let V_ζ, V_χ be as in Definition5.2. Denote $\mathscr{M}$ the Mellin transform. Then one has*

$$\mathscr{M}(V_\zeta), \mathscr{M}(V_\chi) \subset H^2(\Delta_\varepsilon).$$

Proof. We show the case for V_ζ. It is clear that we just need to prove for the monomial term $\widehat{\mathcal{C}^m\mathcal{Z}\eta}, \widehat{\mathcal{C}^{*n}\mathcal{Z}\eta} \in H^2(\Delta_\varepsilon)$.

First, by Lemmas5.55.6, we have $\widehat{\mathcal{C}^m\mathcal{Z}\eta} = \frac{\widehat{\mathcal{Z}\eta}}{(1-s)^m}$, $\widehat{\mathcal{C}^{*n}\mathcal{Z}\eta} = \frac{\widehat{\mathcal{Z}\eta}}{s^n}$. We show that

$$\frac{1}{(1-s)^m}, \frac{1}{s^n} \in H^2(\Delta_\varepsilon).$$

For each $\varepsilon < \sigma < 1-\varepsilon$, there are

$$\begin{aligned}
\frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{|s|^{2n}}ds &\le \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{|s|^{2n}}ds \\
&= \int_{-\infty}^{+\infty} \frac{1}{(\sigma^2+y^2)^n}dy \\
&\le \int_{-\infty}^{+\infty} \frac{1}{(\varepsilon^2+y^2)^n}dy \\
&= \int_{-\infty}^{-1} \frac{dy}{(\varepsilon^2+y^2)^n} + \int_{1}^{\infty} \frac{dy}{(\varepsilon^2+y^2)^n} + \int_{-1}^{1} \frac{dy}{(\varepsilon^2+y^2)^n} \\
&\le \int_{-\infty}^{+\infty} \frac{dy}{\varepsilon^2+y^2} + \int_{-1}^{1} \frac{dy}{\varepsilon^{2n}} \\
&= \frac{\pi}{\varepsilon} + \frac{2}{\varepsilon^{2n}}.
\end{aligned}$$

Hence $\sup\limits_{\varepsilon<\sigma<1-\varepsilon} \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} |\frac{1}{s^n}|^2 ds < \infty$. Therefore, $\frac{1}{s^n} \in H^2(\Delta_\varepsilon)$. Similarly, we have $\frac{1}{(1-s)^m} \in H^2(\Delta_\varepsilon)$. Then by Cauchy-Bunyakovsky-Schwarz inequality

$$\left(\frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} \left|\frac{\widehat{\mathcal{Z}\eta}}{s^n}\right| ds\right)^2 \le \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} |\widehat{\mathcal{Z}\eta}|^2 ds \cdot \frac{1}{i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{ds}{|s^n|^2},$$

which implies $\frac{\widehat{\mathcal{Z}\eta}}{s^n} \in H^2(\Delta_\varepsilon)$. Similarly, $\frac{\widehat{\mathcal{Z}\eta}}{(1-s)^m} \in H^2(\Delta_\varepsilon)$. □

Conjecture 6.3. *The map* $\mathscr{M}^{-1}: H^2(\Delta_\varepsilon) \to L^2(\mathbb{R}_+^\times, dx)$ *is continuous.*

Definition 6.4. Denote $\Omega_{<0}$ the left half-plane of $\mathbb{C}$. Define the PW-transform $\mathscr{S}$ by

$$\mathscr{S}: L^2(\mathbb{R}_+^\times, dx) \to H^2(\Omega_{<0}),\ f(x) \mapsto (\mathscr{S}f)(s) = \int_1^\infty f(\log x)x^{s-1}dx.$$

Theorem 6.5. *The transform* $\mathscr{S}$ *is unitary operator.*

Proof. This is essentially a version of theorem of Paley and Wiener. Let $f(x) \in L^2(\mathbb{R}_+^\times, dx)$. Then $\mathscr{F}f(z) = \int_0^\infty f(x)e^{izx}dx \in H^2(\mathbb{H})$. Let $iz = s$. We have

$$\mathscr{F}f(-is) = \int_0^\infty f(x)e^{sx}dx = \int_1^\infty f(\log x)x^{s-1}dx.$$

Thus $(\mathscr{S}f)(s) = \mathscr{F}f(-is)$ is in $H^2(\Omega_{<0})$. □

Lemma 6.6. *Let* $f \in C^\infty(\mathbb{R}_+^\times)$. *Denote* $F(x) = -xf'(x)$, *where* $f(x)$ *decays rapidly when* $x \to \infty$. *Let* $\widehat{F}(s)$ *be the transform of* $F(x)$ *by*

$$\widehat{F}(s) = \int_1^\infty F(x)x^{s-1}dx.$$

For the operator $\mathcal{C}^*$*, we have*

$$\widehat{\mathcal{C}^*F}(s) = \frac{\widehat{F}(s) - \mathcal{C}^*F(1)}{s}, \quad s \in \mathbb{C}.$$

Denote $G(x) = (xf(x))'$*, where* $f(x)$ *decays rapidly when* $x \to \infty$*. For the operator* $\mathcal{C}$*, we have*

$$\widehat{\mathcal{C}G}(s) = \frac{\widehat{G}(s) + \mathcal{C}G(1)}{1-s}, \quad s \in \mathbb{C}.$$

Proof. First, $\mathcal{C}^*F(x) = \int_x^\infty \frac{F(t)}{t}dt = \int_x^\infty \frac{-tf'(t)}{t}dt = f(x)$. Then $\widehat{\mathcal{C}^*F}(s) = \widehat{f}(s)$. On the other hand,

$$\begin{aligned}
\widehat{F}(s) &= \int_1^\infty F(x)x^{s-1}dx \\
&= -\int_1^\infty x^s f'(x)dx \\
&= -\int_1^\infty x^s df(x) \\
&= f(1) + s\int_1^\infty f(x)x^{s-1}dx \\
&= \mathcal{C}^*F(1) + s\widehat{f}(s).
\end{aligned}$$

Thus $\widehat{\mathcal{C}^*F}(s) = \frac{\widehat{F}(s) - \mathcal{C}^*F(1)}{s}$.

It is easy to see $\mathcal{C}G(x) = f(x)$. Therefore, $\widehat{\mathcal{C}G}(s) = \widehat{f}(s)$. Moreover,

$$\begin{aligned}
\widehat{G}(s) &= \int_1^\infty G(x)x^{s-1}dx \\
&= \int_1^\infty x^{s-1}d(xf(x)) \\
&= -f(1) - \int_1^\infty xf(x)dx^{s-1} \\
&= -\mathcal{C}G(1) + (1-s)\int_1^\infty f(x)x^{s-1}dx \\
&= -\mathcal{C}G(1) + (1-s)\widehat{f}(s).
\end{aligned}$$

Hence, $\widehat{\mathcal{C}G}(s) = \frac{\widehat{G}(s) + \mathcal{C}G(1)}{1-s}$. □

Theorem 6.7. *Each function* $f(s) \in \mathscr{S}(V_\zeta) \subset H^2(\Omega_{<0})$ *is of the form*

$$f(s) = \mathscr{S}\mathcal{Z}\eta(s)\left(\sum_{j=1}^{n}\frac{a_j}{s^j} + \sum_{k=1}^{m}\frac{b_k}{(1-s)^k}\right) + \left(\sum_{j=1}^{n-1}\frac{c_j}{s^j} + \sum_{k=1}^{m-1}\frac{d_k}{(1-s)^k}\right),$$

where n *is some positive integer and* $a_j, b_k, c_j, d_k \in \mathbb{C}$*.*

Similarly, each function $g(s) \in \mathscr{S}(V_\chi) \subset H^2(\Omega_{<0})$ *is of the form*

$$g(s) = \mathscr{S}\mathcal{Z}_\chi\eta_\chi(s)\left(\sum_{j=1}^{n}\frac{a_j}{s^j} + \sum_{k=1}^{m}\frac{b_k}{(1-s)^k}\right) + \left(\sum_{j=1}^{n-1}\frac{c_j}{s^j} + \sum_{k=1}^{m-1}\frac{d_k}{(1-s)^k}\right).$$

Proof. By Proposition5.3, we just need to consider the monomial term $\mathscr{S}(\mathcal{C}^m\mathcal{Z}\eta)$ and $\mathscr{S}(\mathcal{C}^{*n}\mathcal{Z}\eta)$. We prove the case of $\mathscr{S}(\mathcal{C}^{*n}\mathcal{Z}\eta)$, the other one is similar.

First, by Lemma6.6, let $F(x)=\mathcal{Z}\eta(\log(x))$, we have

$$\mathscr{S}\mathcal{C}^*\mathcal{Z}\eta(s)=\frac{\mathscr{S}\mathcal{Z}\eta(s)-\mathcal{C}^*\mathcal{Z}\eta(0)}{s}.$$

Suppose

$$\mathscr{S}(\mathcal{C}^{*n}\mathcal{Z}\eta)=\frac{\mathscr{S}\mathcal{Z}\eta(s)-s\mathcal{C}^*\mathcal{Z}\eta(0)-\cdots-s^{n-1}\mathcal{C}^{*n}\mathcal{Z}\eta(0)}{s^n}.$$

Then

$$\begin{aligned}\mathscr{S}(\mathcal{C}^{*n+1}\mathcal{Z}\eta)&=\frac{\mathscr{S}(\mathcal{C}^{*n}\mathcal{Z}\eta)-\mathcal{C}^{*n+1}\mathcal{Z}\eta(0)}{s}\\&=\frac{\mathscr{S}\mathcal{Z}\eta(s)-s\mathcal{C}^*\mathcal{Z}\eta(0)-\cdots-s^{n}\mathcal{C}^{*n+1}\mathcal{Z}\eta(0)}{s^{n+1}}.\end{aligned}$$

Hence, for $\sum_{j=1}^n a_j\mathcal{C}^{*j}\mathcal{Z}\eta$, we have

$$\mathscr{S}\left(\sum_{j=1}^n a_j\mathcal{C}^{*j}\mathcal{Z}\eta\right)=\mathscr{S}\mathcal{Z}\eta(s)\sum_{j=1}^n\frac{a_j}{s^j}+\sum_{j=1}^{n-1}\frac{b_j}{s^j},$$

where $b_j\in\mathbb{C}$. Then the theorem follows from the above equation. □

Theorem 6.8. *Let η and η_χ be as in (5.1)(5.2). Then for $\mathcal{Z}\eta,\mathcal{Z}_\chi\eta_\chi$, the PW transform $\mathscr{S}\mathcal{Z}\eta(s),\mathscr{S}\mathcal{Z}_\chi\eta_\chi(s)$ are holomorphic functions on $\mathbb{C}$. Moreover,*

$$\mathscr{S}\mathcal{Z}\eta(0)=1,\quad \mathscr{S}\mathcal{Z}\eta(1)=\int_0^\infty\mathcal{Z}\eta(x)e^x dx\neq 0.$$

Proof. We prove the case of η. The other one is similar. First

$$\mathscr{S}\mathcal{Z}\eta(s)=\int_0^\infty\mathcal{Z}\eta(x)e^{sx}dx$$

is holomorphic on the left half-plane. Consider the function $\mathcal{Z}\eta(x)e^{sx}$. Then

$$\begin{aligned}|\mathcal{Z}\eta(x)e^{sx}|&=8\pi\left|\sum_{n=1}^\infty(nx)^2\left(\pi(nx)^2-\frac{3}{2}\right)e^{-\pi(nx)^2}e^{sx}\right|\\&\leq 8\pi\sum_{n=1}^\infty(nx)^2\left(\pi(nx)^2+\frac{3}{2}\right)e^{-\pi(nx)^2+\mathrm{Re}(s)x}\\&\leq 8\pi\sum_{n=1}^\infty(nx)^2\left(\pi(nx)^2+\frac{3}{2}\right)e^{-\pi(nx)^2+\mathrm{Re}(s)nx}\\&=\mathcal{Z}\left(8\pi x^2\left(x^2+\frac{3}{2}\right)e^{-\pi x^2+\mathrm{Re}(s)x}\right).\end{aligned}$$

Since $f(x):=8\pi x^2\left(x^2+\frac{3}{2}\right)e^{-\pi x^2+\mathrm{Re}(s)x}$ is a Schwartz function, then $\mathcal{Z}f(x)$ decay rapidly when $x\to\infty$(see [6, Lem.6.1]). Thus $\mathscr{S}\mathcal{Z}\eta(s)$ is holomorphic on $\mathbb{C}$.

When $s=0$, there is

$$\mathscr{S}\mathcal{Z}\eta(0)=\int_0^\infty\mathcal{Z}\eta(x)dx=1.$$

When $s=1$, there is

$$\mathscr{S}\mathcal{Z}\eta(1)=\int_0^\infty \mathcal{Z}\eta(x)e^x dx \approx 1.92628,$$

where the coarse estimation is obtained by SageMath. The codes are as follows:

```
sun=0
for k in [1..1000]:
  sun+ = 8*pi*exp(x)*k^2*x^2*(pi*k^2*x^2-3/2)*exp(-pi*k^2*x^2)
numerical_integral(sun,0, +Infinity)
```

□

Theorem 6.9. *Let $F_\rho(x)$ be as in (5.3). Then we have*

$$\begin{cases} \mathcal{C}^* F_\rho(x)_\zeta = \frac{1}{\rho}F_\rho(x)_\zeta - \frac{1}{\rho}\mathcal{C}^*\mathcal{Z}\eta(x), & \text{for } \zeta(s) \\ \mathcal{C}^* F_\rho(x)_\chi = \frac{1}{\rho}F_\rho(x)_\chi - \frac{1}{\rho}\mathcal{C}^*\mathcal{Z}_\chi\eta_\chi(x), & \text{for } L(\chi,s). \end{cases} \tag{6.1}$$

Proof. We prove the case for $\zeta(s)$. Since

$$-xF'_\rho(x)=\rho F_\rho(x)+\mathcal{Z}\eta(x),$$

dividing by x , we have

$$-F'_\rho(x)=\rho\frac{F_\rho(x)}{x}+\frac{\mathcal{Z}\eta(x)}{x}.$$

Integrating on the equation, we obtain

$$-\int_x^\infty F'_\rho(t)dt=\rho\int_x^\infty \frac{F_\rho(t)}{t}dt+\int_x^\infty\frac{\mathcal{Z}\eta(t)}{t}dt,$$

that is,

$$\mathcal{C}^*F_\rho(x)=\frac{1}{\rho}F_\rho(x)-\frac{1}{\rho}\mathcal{C}^*\mathcal{Z}\eta(x).$$

□

Theorem 6.10. *Let $\mathscr{S}$ be PW-transform and $F_\rho(x)$ be as in (5.3). Then $\mathscr{S}F_\rho(s)$ is holomorphic as $s=0,1$.*

Proof. We just show the case for $\zeta(s)$. By equation(6.1), we have

$$\mathscr{S}\mathcal{C}^*F_\rho(s)=\frac{1}{\rho}\mathscr{S}F_\rho(s)-\frac{1}{\rho}\mathscr{S}\mathcal{C}^*\mathcal{Z}\eta(s).$$

Take $F(x)=F_\rho(\log x)$ in Lemma6.6. Then we have

$$\frac{\mathscr{S}F_\rho(s)-\mathcal{C}^*F_\rho(0)}{s}=\frac{1}{\rho}\mathscr{S}F_\rho(s)-\frac{1}{\rho}\frac{\mathscr{S}\mathcal{Z}\eta(s)-\mathcal{C}^*\mathcal{Z}\eta(0)}{s}.$$

Therefore,

$$\mathscr{S}F_\rho(s)=\frac{\rho\mathcal{C}^*F_\rho(0)+\mathcal{C}^*\mathcal{Z}\eta(0)-\mathscr{S}\mathcal{Z}\eta(s)}{\rho-s},$$

which implies that $\mathscr{S}F_\rho(s)$ is holomorphic at $s=0,1$ by Theorem6.8. □

Recall the definition of normal families of meromorphic functions. If $z,w\in\mathbb{C}$, the spherical distance is

$$d_S(z,w)=\frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

$$d_S(z,\infty)=\frac{2}{\sqrt{1+|z|^2}}$$

Let $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. A meromorphic function $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is that whose poles are in some discrete closed set of $\mathbb{P}^1_{\mathbb{C}}$.

Definition 6.11. A family $\mathcal{F}$ of meromorphic functions on a domain $D \subseteq \mathbb{C}$ is normal if whenever $\{f_n\}$ is a sequence in $\mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ and $f : D \to \mathbb{P}^1_{\mathbb{C}}$ such that for all compact $K \subseteq D$,

$$\sup_K d_S(f_{n_j}(z) - f(z)) \to 0.$$

Remark 6.12. We allow the function f to be ∞.

In 1979, Gu[8] proved the following well-known normality criterion, which was a conjecture of Hayman[11]. That is the following theorem

Theorem 6.13. *Let $\mathcal{F}$ be a family of meromorphic functions defined in $D \subseteq \mathbb{C}$, and let k be a positive integer. If, for every function $f \in \mathcal{F}$, $f \neq 0, f^{(k)} \neq 1$, then $\mathcal{F}$ is normal.*

Lemma 6.14. *If $\{f_n\}$ is meromorphic on a domain $D \subseteq \mathbb{C}$,*

$$\sup_K d_S(f_n(z) - f(z)) \to 0$$

for all compact $K \subseteq D$, then f is meromorphic on D or $f = \infty$.

Proof. See [17, Lem.1.1]. ∎

Now we can prove a stronger result.

Theorem 6.15. *Let $\overline{V_\zeta}$ (resp.$\overline{V_\chi}$) be the closure of V_ζ (resp. V_χ) in $L^2(\mathbb{R}_+^\times, dx)$. Then $F_\rho(x) \notin \overline{V_\zeta}$ (resp. $F_\rho(x) \notin \overline{V_\chi}$).*

Proof. We only prove the case of $\zeta(s)$. First, $F_\rho(x) \notin V_\zeta$. Suppose there exists a convergent sequence $\{f_n(x)\} \in V_\zeta$ such that $\lim_{n\to\infty} f_n(x) = F_\rho(x)$. Then by Theorem6.5, in $H^2(\Omega_{<0})$, there exists a convergent sequence under norm topology

$$\lim_{n\to\infty} \mathscr{S} f_n(s) = \mathscr{S} F_\rho(s).$$

Then for almost all $s \in \Omega_{<0}$, there exists a subsequence $\{\mathscr{S} f_{n_j}(s)\}$ of $\{\mathscr{S} f_n(s)\}$ converges pointwise to $\mathscr{S} F_\rho(s)$ ([19, Thm.3.12]). From Theorems6.7, 6.13, the sequence $\{\mathscr{S} f_{n_j}(s)\}$ is normal on $\mathbb{C}$. By Lemma 6.14, for some convergent pointwise subsequence of $\{\mathscr{S} f_{n_j}(s)\}$, there exists the limit function

$$f(s) := \lim_{n_j\to\infty} \mathscr{S} f_{n_j}(s)$$

is meromorphic on $\mathbb{C}$.

Since $f(s)$ and $\mathscr{S} F_\rho$ are meromorphic functions and they are a.e. identity in $\Omega_{<0}$, we have $f(s) = \mathscr{S} F_\rho$ for all $s \in \mathbb{C}$. By Theorems6.7,6.8, each $\mathscr{S} f_n(s)$ has at least one pole only at $s = 0, 1$. Hence, the meromorphic function $f(s) = \lim_{n\to\infty} \mathscr{S} f_{n_j}(s)$ has a pole or essential singularity at $s = 0, 1$. However, $\mathscr{S} F_\rho(s)$ is holomorphic at $s = 0, 1$. This is a contradiction. Hence, $F_\rho(x) \notin \overline{V_\zeta}$. ∎

Now we can give the definition of Hilbert–Pólya space.

Definition 6.16. The quotient Hilbert space $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$ (resp. $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\chi}$) is called Hilbert-Pólya space of the operator $\mathcal{C}$ with respect to Riemann zeta function(resp. Dirichlet L-function).

7. The spectrum of $\mathcal{C}$ and $\mathcal{C}^*$ on Hilbert-Pólya space

This section, we prove the Riemann hypothesis for Riemann zeta function and Dirichlet L-function, which is inspired by Connes' work[5], Meyer's paper[15], Li's result[12] and Wu's work[23].

Lemma 7.1. *$\overline{V_\zeta}$ and $\overline{V_\chi}$ are invariant spaces of $\mathcal{C}$ and $\mathcal{C}^*$.*

Proof. First, V_ζ and V_χ are invariant spaces of $\mathcal{C}$ and $\mathcal{C}^*$. Let $\{f_n\}$ be a convergent sequence in $\overline{V_\zeta}$, where $f_n \in V_\zeta$. Denote $\lim_{n\to\infty} f_n = f \in \overline{V_\zeta}$. Since $\mathcal{C}$ is bounded on $L^2(\mathbb{R})_+^\times$, i.e., continuous, we have $\mathcal{C}f = \lim_{n\to\infty} \mathcal{C}f_n \in \overline{V_\zeta}$ under norm topology. Thus $\overline{V_\zeta}$ is an invariant space of $\mathcal{C}$. Similar, it is an invariant space of $\mathcal{C}^*$. The discussion for $\overline{V_\chi}$ is similar. □

Theorem 7.2. *Let ρ be a nontrivial zero of $\zeta(s)$ (resp. $L(\chi, s)$). Then $\frac{1-\rho}{\rho}$ is an eigenvalue of $\mathcal{C}^* - 1$ on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$ (resp. $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\chi}$).*

Proof. We just prove the case for Riemann zeta function $\zeta(s)$. The case for Dirichlet L-function is similar. Let ρ be a nontrivial zero of $\zeta(s)$. Then $1-\rho$ is also a nontrivial zero. By equation(6.1), there is

$$\mathcal{C}^* F_\rho(x) = \frac{1}{\rho} F_\rho(x) - \frac{1}{\rho}\mathcal{C}^*\mathcal{Z}\eta(x).$$

Hence,

$$(\mathcal{C}^* - 1)F_\rho(x) = \frac{1-\rho}{\rho} F_\rho(x) - \frac{1}{\rho}\mathcal{C}^*\mathcal{Z}\eta(x). \tag{7.1}$$

Thus, $\frac{1-\rho}{\rho}$ is an eigenvalue of $\mathcal{C}^* - 1$ on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$. □

Theorem 7.3. *$\mathcal{C}^* - 1$ is a unitary operator on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$ and $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\chi}$.*

Proof. We prove the case for $\zeta(s)$. Since $\overline{V_\zeta}$ is an invariant subspace of $\mathcal{C} - 1$, for each $x \in \overline{V_\zeta}^\perp, y \in \overline{V_\zeta}$, we have

$$\langle (\mathcal{C}^* - 1)x, y\rangle = \langle x, (\mathcal{C} - 1)y\rangle = 0.$$

Hence, $(\mathcal{C}^* - 1)x \in \overline{V_\zeta}^\perp$, i.e., $\overline{V_\zeta}^\perp$ is an invariant subspace of $\mathcal{C}^* - 1$. Then by Theorem5.1, $\mathcal{C}^* - 1$ is a unitary operator on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$. □

Theorem 7.4. *The Riemann hypothesis is true for Riemann zeta function and Dirichlet L-function.*

Proof. By Theorem7.3, the bounded operator $\mathcal{C}^* - 1$ is a unitary operator on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$, whose spectrum is in the unite circle $\{z \in \mathbb{C} : |z| = 1\}$. Therefore, the Riemann hypothesis follows by Theorem7.2. Similarly, it is true for Dirichlet L-function. □

The eigenvalue of Cesàro-Hardy operator is related with the Hilbert space. For example, for the space $L^2[0, 1]$, the set of eigenvalue of Cesàro-Hardy is $\{s \in \mathbb{C} : |s-1| < 1\}$(see [1]). However, the set of eigenvalue of Cesàro-Hardy operator on the Hilbert space ℓ^2 is empty and the set of eigenvalue of the adjoint of Cesàro-Hardy operator on it is $\{s \in \mathbb{C} : |s - 1| < 1\}$(see [3]).

Theorem 7.5. *Let $f(x) \in L^2(\mathbb{R}_+^\times, dx)$ such that its Mellin transform $\widehat{f}(s)$ is analytic function on some strip of $\mathbb{C}$. Then $f(x)$ can not be an eigenvvector of Cesàro-Hardy operator or its adjoint.*

Proof. Let $\mathcal{C}$ be the Cesàro-Hardy operator on $L^2(\mathbb{R}_+^\times, dx)$. Take $f(x) \in L^2(\mathbb{R}_+^\times, dx)$. Suppose

$$\mathcal{C}f = \lambda f,$$

for some $\lambda \in \mathbb{C}$. Then by Lemma5.5, we have

$$\widehat{\mathcal{C}f}(s) = \frac{\widehat{f}(s)}{1-s} = \lambda \widehat{f}(s).$$

Thus $\widehat{f}(s)(\lambda(1-s)-1) = 0$ on some strip. This means $\widehat{f}(s) = 0$. Hence, $f(x) = 0$.

The case for adjoint operator is similar. □

Theorem 7.6. *Let ρ be a nontrivial zero of $\zeta(s)$ or $L(\chi, s)$. Then $\frac{1}{\rho}$ is an eigenvalue of $\mathcal{C}$ and $\mathcal{C}^*$.*

Proof. We just prove the case of $\mathcal{C}^*$. Since $\frac{1}{\rho}$ is an eigenvalue of $\mathcal{C}^*$ on $L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta}$, from the isomorphim

$$L^2(\mathbb{R}_+^\times, dx)/\overline{V_\zeta} \simeq \overline{V_\zeta}^\perp \subset L^2(\mathbb{R}_+^\times, dx),$$

it is also an eigenvalue of $\mathcal{C}^*$ on $L^2(\mathbb{R}_+^\times, dx)$.

We can also prove directly. Notice that

$$\mathcal{C}^* F_\rho(x) = \frac{1}{\rho} F_\rho(x) - \frac{1}{\rho} \mathcal{C}^* \mathcal{Z}\eta(x). \tag{7.2}$$

Let

$$F_\rho(x) = f_\rho(x) + g_\rho(x) \tag{7.3}$$

where $f_\rho(x) \in \overline{V_\zeta}^\perp$, $g_\rho(x) \in \overline{V_\zeta}$. Here $f_\rho(x) \neq 0$, otherwise we have $F_\rho(x) = g_\rho(x) \in \overline{V_\zeta}$, a contradiction.

Putting the equality (7.3) in the equation(7.2), we obtain

$$\mathcal{C}^* f_\rho(x) - \frac{1}{\rho} f_\rho(x) = -\mathcal{C}^* g_\rho(x) + \frac{1}{\rho} g_\rho(x) - \frac{1}{\rho} \mathcal{C}^* \mathcal{Z}\eta(x).$$

Thus $\mathcal{C}^* f_\rho(x) - \frac{1}{\rho} f_\rho(x) \in \overline{V_\zeta}^\perp \cap \overline{V_\zeta}$, we have

$$\mathcal{C}^* f_\rho(x) = \frac{1}{\rho} f_\rho(x).$$

Thus $\frac{1}{\rho}$ is an eigenvalue of $\mathcal{C}^*$ on $L^2(\mathbb{R}_+^\times, dx)$ with eigenvector f_ρ. □

Combining with the property of spectrum of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times, dx)$, we again obtain the Riemann hypothesis

Theorem 7.7. (*Another method*) *The Riemann hypothesis is true for Riemann zeta function and Dirichlet L-function.*

Proof. For each nontrivial zero ρ of zeta function, $\frac{1}{\rho}$ is an eigenvalue of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times, dx)$. Since the spectrum of $\mathcal{C}$ on $L^2(\mathbb{R}_+^\times, dx)$ is the circle

$$\sigma(\mathcal{C}, L^2) = \{z \in \mathbb{C} : |1-z| = 1\},$$

the Riemann hypothesis follows from this result. □

At last, we propose the following conjecture

Conjecture 7.8. *The spectrum of* $\mathcal{C}$ *and* $\mathcal{C}^*$ *on* $L^2(\mathbb{R}_+^\times, dx)$ *except* 0 *are both of point spectrum.*

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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